Persistence in higher dimensions: A finite size scaling study

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We show that the persistence probability $P(t,L)$, in a coarsening system of linear size L at a time t, has the finite-size scaling form $P(t,L)\sim L^{-z\theta}f(t/L^z)$, where θ is the persistence exponent and z is the coarsening exponent. The scaling function $f(x)\sim x^{-\theta}$ for $x\ll 1$ and is constant for large x. The scaling form implies a fractal distribution of persistent sites with power-law spatial correlations. We study the scaling numerically for the Glauber-Ising model at dimension $d=1$ to 4 and extend the study to the diffusion problem. Our finite-size scaling ansatz is satisfied in all these cases providing a good estimate of the exponent θ.

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Persistence decay has been the subject of considerable research activity in recent years. The basic quantity under investigation is the persistence probability $P(t)$, which is the probability that a given stochastic variable with zero mean retains its sign throughout the time interval $[0:t]$. For a large number of systems, it was found that at asymptotic times $t, P(t)\sim t^{-\theta}$, where θ is in general, a dimension-dependent, nontrivial exponent, believed to be unrelated to the other known exponents [1]. The nontriviality of θ is particularly true for spatially extended systems where the time evolution of the stochastic field at one lattice site is coupled to that of its neighbors, making the effective single-site evolution non-Markovian.

In recent times, the spatial aspects of the persistence problem have also come under study. In particular, the spatiotemporal evolution of the set of persistent sites has been studied by several authors. These include the diffusion problem in $d=1$ [2], Ising models in spatial dimension $d=1$ [3–5] and $d=2$ [6], and the generalized q-state Potts model in $d=1$ [7]. It was found that the interplay between persistence decay and the underlying coarsening process leads to dynamical scaling and fractal formation in the spatial distribution of the persistent sites. Such fractal structure has also been reported in an experimental study of breath figures [8].

In the present paper, we propose a scaling form for the persistence probability $P(t,L)$ as a function of the lattice size L and time t. We use physical arguments to motivate the scaling form in the context of the Ising model and show that the scaling reflects the fractal nature and power-law correlations in the spatial distribution of persistent sites. We provide numerical evidence for its validity through simulations in spatial dimension $d=1$ to 4. The analysis is further extended to the diffusion problem where approximate analytic theories have been used to predict θ in all dimensions. We argue that fractal formation in diffusion should take place in all dimensions and provide supportive results from simulations.

Let us consider the Ising model in a d-dimensional geometry of linear size L. We start from an initial random configuration and quench the system, say, to the temperature $T=0$. As a result, the spins evolve in time following the Glauber dynamics, lowering the total energy of the configuration in the process. In the course of time, domains of positive and negative spins form, with characteristic length scale $\xi(t)$ growing as a power law in time i.e., $\xi(t)\sim t^{1/z}$, where z is the dynamical exponent for the coarsening process [9]. The fraction of persistent spins decays as a power of time: $P(t,L)\sim t^{-\theta}$ as long as $t\ll t^{*}\sim L^{z}$. For $t\gg t^{*}$, the domain cannot grow any further because of the finite system size and persistence probability stops decaying, attaining a limiting value $P(\infty,L)\sim L^{-z\theta}$. This happens as long as

$$\frac{z\theta}{d}<1. \tag{1}$$

For $z\theta>d$, persistence probability will decay to zero for any lattice size L. Also we assume that there is no ''blocking,'' whereby a finite fraction of spins never flip, leading to a limiting value P_∞ independent of finite-size effects. Such a situation is believed to occur in Ising model for dimensions $d>4$ [10] and in disordered systems [11].

The above behavior of the persistent fraction $P(t,L)$ for finite lattice sizes can be summarized in the following dynamical scaling form:

$$P(t,L)=L^{-z\theta}f(t/L^{z}), \tag{2}$$

where the scaling function $f(x)\sim x^{-\theta}$ for $x\ll 1$ and $f(x)\to$constant at large x. Similar finite-size scaling ideas have been used in a previous work in the context of the global persistence exponent for nonequilibrium critical dynamics [12].

The finite-size scaling form given by Eq. (2) implies the presence of scale-invariant spatial correlations in the system, characteristic of fractals. To show this, we consider the two-point correlation function $C(r,t)$, which we define as the probability of finding a persistent spin at a distance r from another persistent spin. For a d-dimensional system, $C(r,t)$ satisfies the normalization condition $\int_0^L C(r,t)d^dr=L^dP(t,L)$. After substituting Eq. (2), this becomes

$$\int_0^L C(r,t)r^{d-1}dr\sim L^{d-z\theta}f(t/L^{z}). \tag{3}$$

Let us rewrite this equation in terms of a new function $F(a,b)=a^{z\theta}C(a,b)$ and dimensionless variables $x=r/L$ and $\tau=t/L^{z}$:

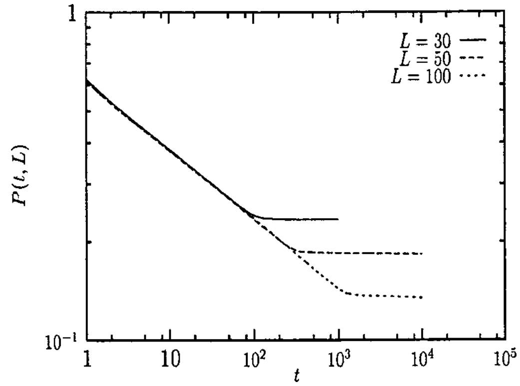

FIG. 1. The persistence probability $P(t,L)$ is plotted against time t (measured in MC steps) for three different lattice sizes L in the $d=2$ Glauber Ising model.

$$\int_0^1 F(Lx, L^z\tau) x^{d-1-z\theta} dx \sim f(\tau). \tag{4}$$

Since the righthand side (RHS) of the equation has no explicit L dependence, the left-hand side (LHS) should also be likewise. This is possible only if $F(a,b)=g(ba^{-z})$, where $g(\eta)$ is given by the integral relation

$$\tau^{d/z-\theta}\int_\tau^\infty \eta^{\theta-(1+d/z)} g(\eta) d\eta \sim z f(\tau). \tag{5}$$

Using the above equation, the limiting behavior of the function $g(\eta)$ for small and large values of the argument could be deduced from the known behavior of the function $f(\tau)$. Consider $\tau \gg 1$, where $f(\tau)$ is constant. From Eq. (5), this implies that $g(\eta)$ is constant for large η. In the other extreme of $\tau \ll 1$, $f(\tau) \sim \tau^{-\theta}$. We split the integral in Eq. (5) as $\int_\tau^\infty = \int_\tau^\alpha + \int_\alpha^\infty$ and note that $g(\eta)$ is constant in the second integral for sufficiently large α. The second integral vanishes as $\tau^{d/z-\theta}$ as $\tau \to 0$, whereas the RHS diverges as $\tau^{-\theta}$. This can be consistent only if the first integral diverges as $\tau^{-\theta}$, which would imply that $g(\eta) \sim \eta^{-\theta}$ as $\eta \to 0$. This leads to the following dynamical scaling form for $C(r,t)$:

$$C(r,t) = r^{-z\theta} g\left(\frac{t}{r^z}\right). \tag{6}$$

For small separations $r \ll t^{1/z}$, this scaling form implies scale-free correlations, i.e., $C(r,t) \sim r^{-z\theta}$, characteristic of a fractal with fractal dimension $d_f = d - z\theta$. On the other hand, over larger length scales, $C(r,t) \sim t^{-\theta}$, which is indicative of the absence of any spatial correlations. This scaling description was introduced by us [4,5] in the context of $A + A \to \varnothing$ model, and later verified numerically in a two-dimensional Ising model [6] also.

To check the finite-size scaling form given by Eq. (2), we simulate Ising spin systems of various sizes in spatial dimension $d=1$ to 4. Starting from a random initial configuration, the spins are quenched to zero temperature and are updated sequentially using the Glauber updating rule by which a spin is always flipped if the resulting energy change $\Delta E < 0$, never flipped if $\Delta E > 0$, and flipped with probability $\frac{1}{2}$ if $\Delta E = 0$. One MC time step was counted after every spin in the lattice was updated once. The persistence probability at any time t was determined as the fraction of spins that did not flip even once until time t since the time evolution started. The data is averaged typically over 1000 starting random configurations for small L and low d and over 50 starting configurations for large L and high d.

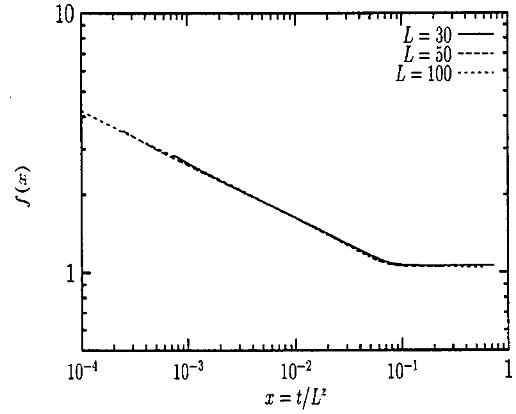

FIG. 2. Same as Fig. 1, except that the scaling function $f(x) = L^{z\theta}P(t,L)$ is plotted against the dimensionless scaling variable $x = t/L^z$. The data for different L values were found to collapse well to a single curve for $\theta = 0.21$ and $z = 2.12 \pm 0.05$.

For $T=0$ Glauber dynamics of the Ising model, the persistence exponent θ is exactly known to be 3/8 in $d=1$ [13]. In higher dimensions, simulations predict $\theta \simeq 0.22$ ($d=2$) [10,14,15] and $\theta \simeq 0.16$ ($d=3$) [10]. In our finite-size scaling analysis of the simulation data, we adopt the following procedure. For $d=1$, 2, and 3, we fix θ at its known value and adjust z to find the value that gives the best data collapse. In all cases, we find $z \simeq 2$, which is the accepted value of the coarsening exponent for nonconserved scalar models [9]. (In $d=3$ Glauber dynamics, a slower $t^{1/3}$ coarsening has been observed before [16]. This is presumably due to lattice effects, but we have not seen any signature of this effect in our simulations.) In $d=4$, on the other hand, we fix z at 2, and adjust θ to collapse the data to a single curve. The results are displayed in Figs. 1–4.

In $d=4$, we find that for $z=2$, $\theta = 0.12 \pm 0.02$ gives reasonably good data collapse over the time scales and system sizes studied. Figure 4 shows the scaled data in $d=4$. It may

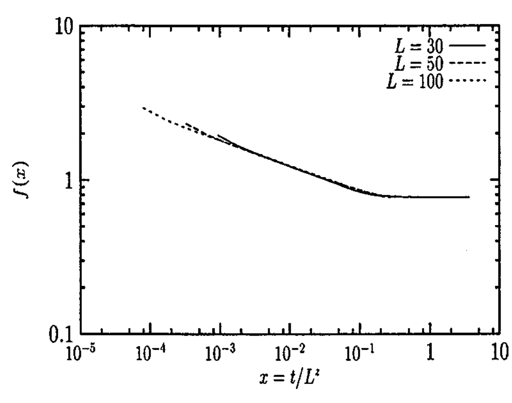

FIG. 3. The scaling function $f(x) = L^{z\theta}P(t,L)$ is plotted against the dimensionless scaled time $x = t/L^z$ for three L values in the $d=3$ Glauber Ising model. The observed data collapse has been obtained for $z = 2.05$ and $\theta = 0.166$.

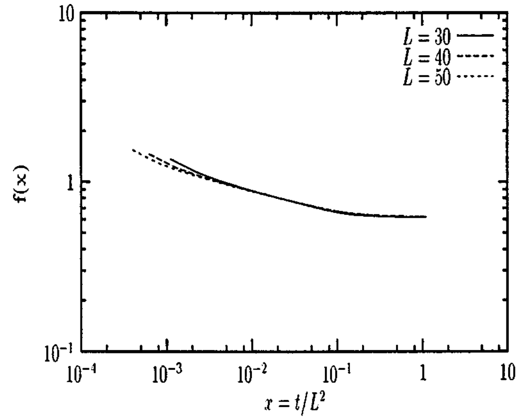

FIG. 4. The figure shows the scaled probability plotted against the dimensionless scaled time in the $d=4$ Glauber Ising model. We have fixed $z=2$, and find that $\theta=0.12\pm0.02$ gives the best data collapse.

be mentioned that in $d=4$, earlier simulations had suggested that the persistence decay might be slower than a power law, and perhaps logarithmic [10]. However, the agreement of our data with the scaling form Eq. (2) suggests that persistence follows a power-law decay in $d=4$ also. For $d>4$, blocking of spins has been shown to lead to a limiting value of $P(t,L)$ as $t\to\infty$, which is independent of L [10]. We could simulate only small lattice sizes for $d=5$ from which we cannot make any conclusive remark at this stage.

In the diffusion problem, we have a scalar field $\phi(\mathbf{x},t)$ evolving according to the diffusion equation. The initial values $\phi(\mathbf{x},0)$ are taken to be independent random variables with zero mean.

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}=\nabla^2\phi(\mathbf{x},t),$$
$$\langle\phi(\mathbf{x},0)\phi(\mathbf{x}',0)\rangle=\delta^d(\mathbf{x}-\mathbf{x}'). \tag{7}$$

For this problem, it has been shown using approximate analytic theories [17–19] that $P(t)\sim t^{-\theta}$ in all dimensions. The predicted exponent values in low dimensions were in good agreement with simulation results. The exponent was found to increase with dimension, and has been suggested to have the asymptotic value $\theta(d)\simeq\alpha\sqrt{d}$ as $d\to\infty$.

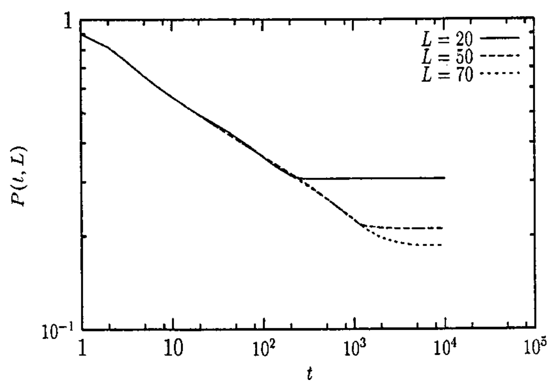

FIG. 5. The persistence probability $P(t,L)$ is plotted against time t (measured as the number of MC steps) for three different lattice sizes L in the $d=2$ diffusion problem.

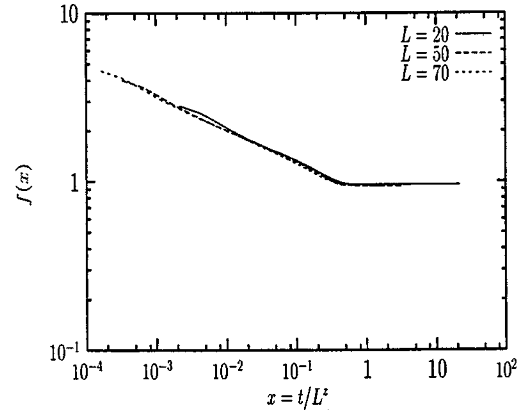

FIG. 6. Same as Fig. 5, except that the scaling function $f(x)=L^{z\theta}P(t,L)$ is plotted against the dimensionless scaling variable $x=t/L^z$. The data for different L values were found to collapse well to a single curve for $\theta=0.186$ and $z=2.05\pm0.04$.

The constant α has been estimated to be $\simeq0.14$ [17,18] and $\simeq0.18$ [19] by different authors. For $d=1$, 2, and 3, the exponent values are found to be $\theta\simeq0.12$, 0.18, and 0.23, respectively.

To simulate Eq. (7) numerically, we use the finite difference Euler discretization scheme on cubic lattices of L^d sites [17,18]:

$$\phi(\mathbf{x},t+\Delta t)=\phi(\mathbf{x},t)+a\left[\sum_{\mathbf{x}'}\phi(\mathbf{x}',t)-2d\,\phi(\mathbf{x},t)\right], \tag{8}$$

where $\mathbf{x}'$ runs over all the $2d$ nearest-neighbor lattice sites of $\mathbf{x}$ in the cubic lattice and $a=\Delta t/(\Delta x)^2<1/2d$ for stability of the discretization scheme. We have taken $a=1/4d$ in our simulations as this value has been observed to provide the fastest approach to the asymptotic regime [17].

For the diffusion problem, simple scaling arguments suggest that the dynamical exponent $z=2$ in all dimensions. In all dimensions studied, we found excellent scaling collapse with $z\simeq2$ and the θ values quoted above. Upon substitution of the exponent values into Eq. (1), it can be easily seen that the condition for fractal formation is satisfied for $d=1$, 2, and 3. For $d=1$, this has already been confirmed by an earlier numerical study [2]. Our results for the persistence probability and the scaling function for three different lattice sizes in $d=2$ is displayed in Figs. 5 and 6.

It is also possible to extrapolate these results to the $d\to\infty$ limit using the asymptotic form suggested for θ. We see that in this limit, the LHS of Eq. (1) vanishes as $1/\sqrt{d}$, leading us to conjecture that fractal formation persists in all dimensions for the diffusion problem.

In conclusion, we have proposed a finite-size scaling ansatz for the persistence probability in a coarsening system. The scaling form corresponds to the fractal structure and dynamic scaling characterizing the spatio-temporal evolution of the persistent set. We check the scaling form numerically for the Glauber-Ising model and for the diffusion problem. Finite-size scaling enables us to study persistence reliably in higher dimensions. Our results agree with the known values of θ in the case of the Ising model (from $d=1$ to 3) and in the diffusion problem (we have checked up to $d=3$). For the

$d=4$ Ising model, we find the signature of algebraic decay of persistence with $\theta \simeq 0.12$, in contrast with what had been reported earlier [10].

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